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MATHEMATICAL MONTHLY.

OCTOBER, 1859.

EDITED BY

J. D. RUNKLE, A.M., A.A.S.

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1859.

Those having taken our second Prize for Solutions may have Vol. II., instead of a bound copy of Vol. I., if preferred. Please notify us how to send the bound copies of Vol. I. . . . . Those wishing to have Vol. I. substantially bound in turkey back and corners, and marbled paper sides, for 87½ cents, will send their numbers to Sever & Francis, Cambridge. . . . . All remittances, and all communications pertaining to Vol. I., should be addressed to the Editor; but every thing not relating to the editorial department of Vol. II., should be addressed to the Publishers, unless it should be more convenient to address the Editor. . . . . In sending the first number of a new volume to the subscribers and friends of the Mathematical Monthly, we earnestly solicit the continuance of their support. Both Editor and Publishers, as will be seen in Editorial Items, stand pledged to make the Monthly every way worthy of continued favor. We feel certain that, to the teachers of the mathematics all over the country, we do not appeal in vain; for if it shall fail to benefit them, then it will in a great measure fail to accomplish the main end of its establishment — the elevation of the standard of mathematical learning in this country. During the coming year we intend to give more attention to the elementary and educational character of the Monthly; and to this end we earnestly invite practical teachers and others to send us articles proper for this department. We hope, also, that teachers will not fail to call the attention of their pupils to the Prize Problems. (See Editorial Items.) The first and second of these Problems, for the best solutions of which, Messrs. Ivison & Phinney offer a copy of the Monthly, are quite easy, and may be made the means of exciting a deep interest in the study of mathematics. We beg of teachers to encourage their pupils to solve these problems, and send us the solutions. Without such coöperation, the prizes must fail of their legitimate effect. . . . . To college students and all those who, during the past year, have competed for our Prizes, but little need be said. We refer to the pages of the Monthly containing the Prize solutions; and feel that, under all the circumstances, we may safely ask of teachers and students not only to aid us in making the prize department valuable and instructive, but also share with us the burden of expense, by extending our circulation. We depend upon our friends, and ask them to coöperate with Editor and Publishers in making the Monthly all that can be desired. No one will fail to remember that it can never rest on the broad basis of a merely literary journal, and that however well it may be conducted, it will always need the kindly aid of friends to make its merits, and claims to support, known. . . . . The Editor takes this opportunity, in behalf of all the friends of the Monthly, to sincerely thank Mr. John Bartlett, for the generous interest he has taken in its publication, and the valuable aid he has rendered. An early number will contain a fine portrait, on steel, of Dr. Bowditch, with a complete list of all his writings.

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THE

MATHEMATICAL MONTHLY.

Vol. II. ... OCTOBER, 1859 ... No. I.

PRIZE PROBLEMS FOR STUDENTS.

I. Solve the equations

$$\sqrt{x} - 4 = \frac{259 - 10x}{4 + \sqrt{x}}$$

$$\frac{10}{x} - \frac{14 - 2x}{x^2} = 2\tfrac{4}{9}.$$

II. In any plane triangle, prove that the sines of the angles are inversely as the perpendiculars let fall from them upon the opposite sides.

III. Having given the diagonals of a quadrilateral inscribed in a given circle to determine its sides geometrically, when the diagonals intersect each other at right angles.

IV. Given

$$(1)\ xw + yz = ab,\ (2)\ xy + zw = ad$$

$$(3)\ xz + yw = bd,\ (4)\ x^2 + w^2 = y^2 + z^2;$$

to find the values of x, y, z, and w

V. If, in a plane or spherical triangle, A, B, C denote the angles, and a, b, c the opposite sides respectively; if r, ϱ denote the radii

of the circumscribed and inscribed circles, and δ the distance between the centres of these circles; then in the plane triangle

$$*\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C$$

and in the spherical triangle

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C \frac{\cos \delta}{\cos r \cos \varrho}.$$

The solution of these problems must be received by the first of December, 1859. [See Editorial Items.]

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. IX., Vol. I.

The first Prize is awarded to William C. Cleveland, of the Lawrence Scientific School, Cambridge, Mass.

The second Prize is awarded to John W. Jenks, Senior Class, Columbia College, N. Y.

Prize Solution of Problem I.

by asher b. evans, madison university, hamilton, n. y.

"In a right-angled triangle, having given the difference between the base and perpendicular, and also the difference between the hypothenuse and base; to construct the triangle geometrically."

On the indefinite line NE take $AM = a =$ the difference between the base and perpendicular, and erect the perpendicular MT. Also, with A as a centre, and $AO = b + a =$ he difference

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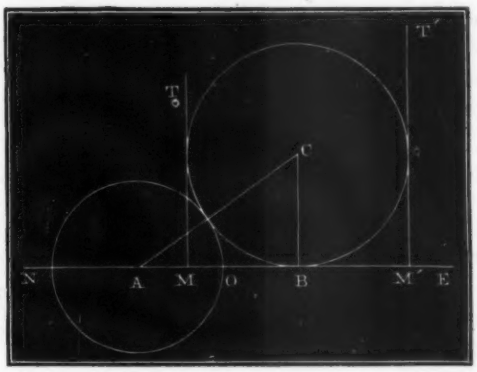
* This analogy, pointed out by D'Arrest, is only one of a number of interesting ones given by Professor Chauvenet, in Gould's Astronomical Journal, Vol. III., page 50, without demonstration. We propose to give them from time to time in our lists of Prize Problems, as most valuable exercises in Trigonometry.

between the hypothenuse and perpendicular as radius, describe a circle. *Now if C be the centre of a circle tangent to this circle, and also to the sides of the rightangle $T\,M\,E$ then will $A\,B\,C$ be the required triangle. For, $A\,B = B\,C + A\,M = B\,C + a$, and $A\,C = B\,C + A\,O = B\,C + b + a = A\,B + b$.



Prize Solution of Problem II.

BY ASHER B. EVANS, MADISON UNIVERSITY, HAMILTON, N. Y.

"In a right-angled triangle, having given the sum of the base and perpendicular, also the sum of the hypothenuse and base; to construct the triangle geometrically."

Let x be the perpendicular, $a - x$ the base, and $b - a + x$ the hypothenuse. On the indefinite line NE take $AM' = a$, and draw the perpendicular $M'\,T'$. Also, with A as a centre, and $AO = b - a$ as a radius, describe a circle. If C be the centre of a circle tangent to this circle, and also to the sides of the right angle $A\,M'\,T'$, then will $A\,B\,C$ be the required triangle. For, $A\,B = A\,M - B\,C = a - x$, and $A\,C = A\,O + B\,C = b - a + x$.

Second Solution of Problems I. and II.

BY GEORGE A. OSBORNE, JR., LAWRENCE SCIENTIFIC SCHOOL.

Problem I. Denote by a the difference between the hypothenuse and base, and by b the difference between the perpendicular and

---

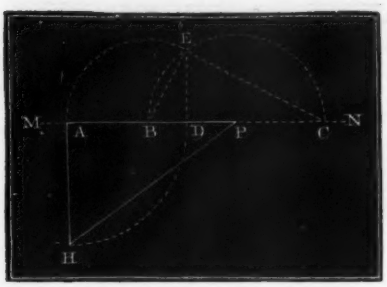
* To draw a circle tangent to a given circle, and to the sides of a rightangle, we may employ Problem 8, in a Memoir on the "Tangencies of Circles," by Major Benjamin Alvord (Smithsonian Contributions to Knowledge); or a Theorem of Professor H. A. Newton, found on page 242 of the Mathematical Monthly, Vol. I.

base; also, let $x =$ base, $y =$ perpendicular; therefore, $\sqrt{x^2+y^2} =$ hypothenuse.

There may be two cases, according as the perpendicular is longer, or shorter, than the base.

For the first case, we have $\sqrt{x^2+y^2} - x = a$, and $y - x = b$; by eliminating y, we readily obtain $x = a - b \pm \sqrt{2a(a-b)}$; rejecting the negative sign before the radical, as giving a negative value to x, we have $x = a - b + \sqrt{2a(a-b)}$ and $y = x + b$.

To construct these values of x and y: From any point P on the line, MN, with radius $PB = a$, describe a semicircle. Lay off $PD = b$, erect the perpendicular DE, make $AB = BE$, and AH equal and perpendicular to AD. Joining H with P forms the required triangle, PAH.



For $AB = BE = \sqrt{BD \cdot BC} = \sqrt{2a(a-b)}$. Therefore, $AH = AD = BD + AB = a - b + \sqrt{2a(a-b)} = x$, and $AP = AD + PD = x + b = y$.

If $b > a$, the value of x will be imaginary, and the construction impossible.

For the second case, $x - y = b$, or $y - x = -b$; hence, if in the expressions for x and y, in the first case, we make b negative, they will apply for the second case; namely,

$$x = a + b + \sqrt{2a(a+b)} \text{ and } y = x - b.$$

If $b < a$, the triangle may be constructed as in the first case, provided we make the same construction with respect to C and CE, as was there made with respect to B and BE.

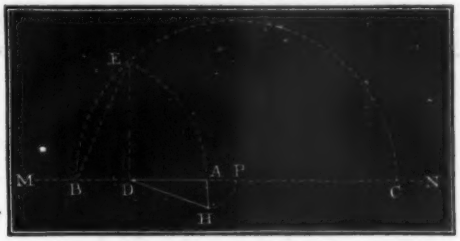
If $b > a$, the semicircle must be described on $a + b$ as a diameter, instead of on $2a$, as before.

PROBLEM II. Denote by a the sum of the hypothenuse and base, and by b the sum of the base and perpendicular; also let $x =$ base, $y =$ perpendicular; therefore, $\sqrt{x^2+y^2} =$ hypothenuse.

Hence, $\sqrt{x^2+y^2}+x=a$, $x+y=b$. Eliminating y gives $x=-(a-b)\pm\sqrt{2a(a-b)}$; rejecting the negative sign before the radical as giving a negative value to x, we have

$$x=-(a-b)+\sqrt{2a(a-b)} \text{ and } y=b-x.$$

To construct these values of x and y: From any point P on the line MN with radius $PB=a$, describe a semicircle. Lay off $PD=b$, erect the perpendicular DE, make $BA=BE$ and AH equal and perpendicular to AP. Joining H with D forms the required triangle DAH.



For $BA=BE=\sqrt{BD.BC}=\sqrt{2a(a-b)}$ $\therefore$ $AD=BA-BD$ $=\sqrt{2a(a-b)}-(a-b)=x$. $AH=AP=PD-AD=b-x=y$.

If $x>b$, or $\sqrt{2a(a-b)}>a$, the value of y will be negative, and the construction impossible; hence, $\sqrt{2a(a-b)}<a$, or $BA<BP$, and the point A always falls between B and P. The expression $\sqrt{2a(a-b)}<a$ is equivalent to $b>\frac{1}{2}a$.

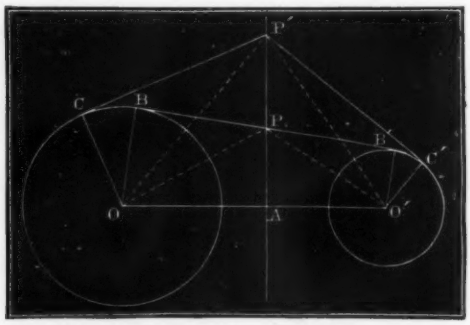
PRIZE SOLUTION OF PROBLEM III.

BY GEORGE A. OSBORNE, JR., LAWRENCE SCIENTIFIC SCHOOL.

"The four tangents, which are common to two circles which do not intersect, and are terminated at their points of respective contact, have their middle points on the radical axis of the two circles."

Through P, the middle point of the common tangent BB', draw $P'A$ perpendicular to OO'. From any point P' of this line draw the tangents $P'C$ and $P'C'$; draw also PO, PO', $P'O$, and $P'O'$. Then we have $P'O^2-OC^2=P'C^2$ and $PO^2-OB^2=PB^2$.

Therefore, by subtraction $P'O^2 - PO^2 = P'C^2 - PB^2$. Similarly we find $P'O'^2 - PO'^2 = P'C'^2 - PB'^2$. Also, $P'O^2 - PO^2 = P'A^2 - PA^2 = P'O'^2 - PO'^2$. $\therefore P'C^2 - PB^2 = P'C'^2 - PB'^2$. But $PB = PB'$; $\therefore P'C = P'C'$. Hence, if from any point of the line $P'A$ tangents be drawn to the two given circles, they will be equal. *$P'A$ is therefore the radical axis of the two circles, and will bisect the common tangents.

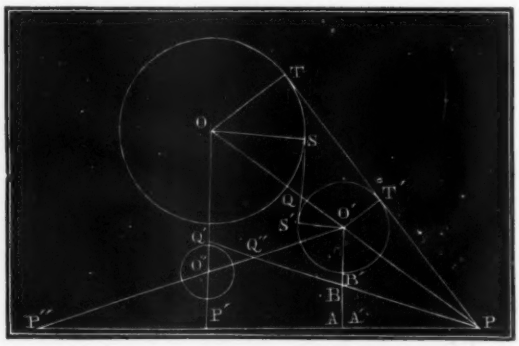


Prize Solution of Problem IV.

BY JOHN W. JENKS, SENIOR CLASS, COLUMBIA COLLEGE, N. Y.

"The external centres of similitude of three circles, taken successively two and two, all lie in one straight line; and each of them is situated in a right line with two of the internal centres of similitude."

In any two circles (O, O') the distances $(OP, O'P)$ from the centres to the external centre of similitude (P) are by similar triangles $(OTP, O'T'P)$ proportional to the radii of the two circles. By like similar triangles $(OSQ, O'S'Q)$ the distances from the centres to the internal centre of similitude are also seen to be proportional to the radii.



* This Problem is a special case of Proposition 5, on page 270, of the June number of the MONTHLY, as was remarked by nearly all the contributors.

Let O O', O'' be the three given circles of the problem, and denote their radii by R, R', R'' respectively; also let P be the external and Q the internal centre of similitude of O and O', P' and Q' of O and O'', and P'' and Q'' of O' and O''. Then will P, P' and P'' lie in one straight line, as also Q', Q'' and P, also Q, Q'' and P', and Q, Q' and P''.

For, draw the lines $O\,O'P$, $O\,O''\,P'$ and $O'\,O''\,P''$, also $P''\,P'$ and $P\,P'$. Now, drawing $O'\,A'$ parallel to $O\,O''\,P'$ to meet $P''\,P'$ produced in A' we have by similar triangles $O'A' : O''P' = O'P'' : O''P''$ $= R' : R''$. $\therefore\ O'A' = \frac{O'\,P' \times R'}{R''}$. Drawing also $O'A$ parallel to $O\,O''P'$ to meet $P\,P'$ in A, we have, by similar triangles, again, $O'A$: $O\,P' = O'\,P : O\,P = R' : R\ \therefore\ O'\,A = \frac{O\,P' \times R'}{R}$. But $O\,P'$: $O''\,P' = R : R''$, or $O\,P' : R = O''\,P' : R''\ \therefore\ \frac{O\,P' \times R'}{R} = \frac{O''\,P' \times R'}{R''}$. $\therefore\ O'\,A = O'\,A'$. Hence the two lines $P''\,P'$ and $P\,P'$, having two points in common (P' and A), coincide throughout their whole extent, and form one and the same straight line.

In a precisely similar manner, by drawing the line $P\,Q''$ and $Q'\,Q''$ and drawing $O'B$ to $P\,Q'$ parallel to $O\,O''\,P'$, and $O'B'$ to $Q'\,Q''$, produced, parallel to the same, we have $O'B : O\,Q' = O'\,P : O\,P = R' : R$. $\therefore\ O'B = \frac{O\,Q' \times R'}{R}$; also $O'\,B' : O''\,Q' = O'\,Q'' : O''\,Q' = R' : R''$. $\therefore\ O'B' = \frac{O''\,Q' \times R'}{R''}$.

But $O\,Q' : O''\,Q' = R : R''$. $\therefore\ O\,Q' : R = O''\,Q' : R''$. $\therefore\ O'B = O'B'$. Hence $P\,Q'$ and Q' are in the same straight line. In the same way it may be proved that $P'\,Q''$ and Q, and also $P''\,Q'$ and Q, are in the same straight line with one another.

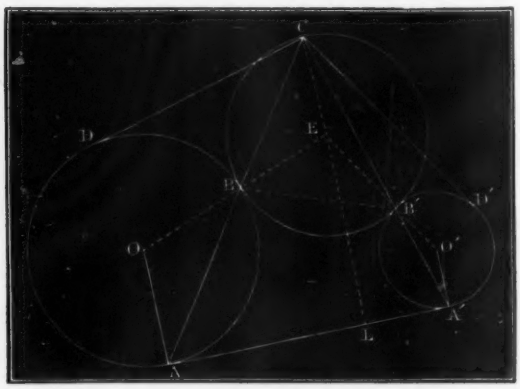
Prize Solution of Problem V.

BY WILLIAM C. CLEVELAND, LAWRENCE SCIENTIFIC SCHOOL.

"Let two circles be touched respectively by a single straight line $A\,A'$ in A and A',

and by a single circle in $BB'C$ in B and B'; if the straight line and the circle touch in the same manner the two circles, the point C of the meeting of AB and $A'B'$ will lie on the circumference of the circle $BB'C$ and on the radical axis of the two other circles."

Let E be the centre of the circle $BB'C$, and O and O' the centres of the two other circles. The line OE passes through B, and $O'E$ through B'. OA and $O'A'$ are parallel, since they are both perpendicular to AA'. Through E draw EL parallel to OA, and suppose AB produced to meet EL produced in C, and $A'B'$ to meet in C'. The similar triangles AOB and BCE give $\frac{EC}{EB} = \frac{AO}{OB} = 1$, and $A'O'B'$ and $B'C'E$ give $\frac{EC'}{EB'} = \frac{O'A'}{O'B'} = 1$. $\therefore EC = EB = EB' = EC'$.



Therefore C and C' are the same point on the circumference of $BB'C$.

Draw the tangents CD and CD'; then

$$\frac{CD^2}{CD'^2} = \frac{CA.CB}{CA'.CB'} = \frac{CA.2EB\cos ECB}{CA'.2EB'\cos ECB'} = \frac{CA\cos ECB}{CA'\cos ECB'} = \frac{CL}{CL} = 1.$$

Therefore $CD = CD'$, and C is on the radical axis of the circles O and O'.

Chauncey Wright.
Truman Henry Safford.

Note. — Prof. Winlock has been ordered to the Naval Academy at Annapolis, Maryland, as Professor of Mathematics.

MECHANICAL CONSTRUCTION OF THE AREA OF A CIRCLE.*

By Chauncey Smith, of the Suffolk Bar, Boston.

Proposition. *The area of a circle is equal to the rectangle contained by its semicircumference and semidiameter.*

Children usually find this proposition in their arithmetics, in the form of a rule, before they have any knowledge of geometry; and I therefore propose the following simple, graphic method of demonstration.

Cut a circle from a piece of card, and bisect it as in Fig. 1. Divide each half into an equal number of sections, cutting from the centre, *C*, but leaving them all joined together at the circumference. Next,

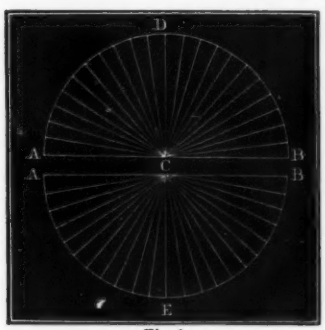


Fig. 1.

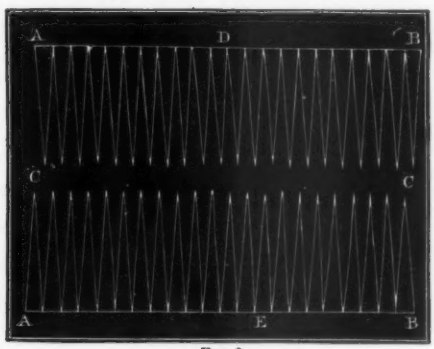


Fig. 2.

draw the points *A* and *B*, as nearly as may be, into straight lines with *D* and *E*, as in Fig. 2. Lastly, slide the halves together as

* This graphic method may be exhibited equally well upon the blackboard, by drawing the figures, and showing how one is made from the other; and it will be found of great advantage in those cases in which the pupil is called upon to work by an arbitrary rule, as in most of the arithmetics. To the same end, the rectangle may be divided into squares, and the pupil made to see that if one of these squares is taken as the unit of surface, that the surface of the whole rectangle will be measured by the product of its base by its altitude; that is, this product will show how many such units the rectangle contains. Indeed, the

in Fig. 3. If the number of sections is very great, ADB and AEB in Fig. 2 will be very nearly straight lines, and Fig. 3 will differ but little from a rectangle. Its surface is clearly the same as that of the circle out of which it was made; but its base and altitude are the semi-circumference and semidiameter of the circle, and their product is the measure of its area, which is therefore the measure of the area of the circle.

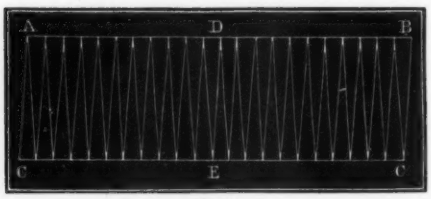


Fig 3.

ON THE PRIME SEVENTH AS AN ESSENTIAL ELEMENT IN THE MUSICAL SYSTEM.

By Henry Ward Poole, Engineer, Boston.

It is now ten years, since, by original investigation in the mathematical, mechanical, and practical departments of music, I was led to the belief that this science has a solid foundation in the relations of numbers, and that all the supposed impossibility of Just Intonation and the necessity of Temperament, have their origin only in the short-sightedness of the theorist, and the unskilfulness of the practitioner.

Having settled upon the rule that musical ratios must not exceed a certain limit of simplicity (the limit to be determined by the

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areas of all the surfaces investigated in plane geometry may be exhibited to the eye by simple construction; and quite young children may readily be taught to understand as well as make the constructions for themselves. — Ed.

Since receiving the above, the same construction has been sent us by R. C. Matthewson, Esq., of San Francisco, California; and we also find it in Lund's Geometry, an English book of recent date.

ability of the ear to appreciate them), it was stated * that those ratios only were admissible which were derived from the prime numbers 2, 3, 5, and 7. That the three lower primes 2, 3, and 5 belong to the musical system has been universally admitted; but no one, before myself, so far as I know, has made this claim for the prime seven.†

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* American Journal of Science, Second Series, Vol. IX. pp. 68, 199.

† I do not wish to conceal the fact, that even now the principle of just intonation (or the possibility, in theory or practice, of exact fifths, thirds, &c.) is denied by high mathematical authorities. Sir J. F. W. HERSCHEL, in his treatise on Sound, declares, that "singers, violin players, and all others who can pass through every gradation of tone, must all temper, or they could never keep in tune with each other or themselves." [The work of HERSCHEL not being at hand, this extract is copied from the treatise on *Sound*, by Professor BENJAMIN PEIRCE, of Cambridge, who has reproduced (with his indorsement, it is presumed,) these and like views of HERSCHEL.] By a late letter from Sir JOHN HERSCHEL, dated Collingwood, June 14th, 1859, addressed to the Musical Pitch Committee, at the Society of Arts, he evinces his continued belief in Temperament as inherent in music, and his opinion that this temperament gives some peculiar character to the different signatures or keys in music in general. He says, in regard to the concert pitch:—

"All are desirous that when once lowered, it should be kept from rising [1] again, to which there is a continual tendency arising from a distinct natural cause inherent in the nature of harmony; namely, the excess (amounting to about eleven vibrations in ten thousand) of a perfect fifth over seven-twelfths of an octave, which has to be constantly contended against in upward modulations, whenever violins or voices are not kept in check by fixed instruments. But perhaps all are not aware that the evil of fine ancient compositions having thus been rendered impracticable to singers in their original normal key involves the sacrifice of the adaptation of the peculiar character of the key (a character intended and felt by the composer), and the substitution of a totally different incidence of the temperament [2] in the series of notes in the scale, and goes therefore to mar the intended effect, and injure the composition, as much as an ill-chosen tone of varnish would damage the effect of a fine Titian."

1. There is nothing better to test the "natural tendency" in this respect than a good glee-club without accompaniment. If a high pitch is taken and they are fatigued, the pitch will gradually fall. If they start with too low a key-note and are in good spirits, the tendency will be to rise to the better pitch. It does not appear that temperament affects the concert pitch.

2. Observe the same glee-singers. They sing in every key with the same relative intervals, and *do not* use a "different incidence of temperament," in different keys. Did any composer of glees wish such temperament? If so he should indorse his score something after this manner: "Four flats, equal temperament" (as the composers of fugues for the organ have actually done;) or "Four flats, with a great wolf in A flat, and a whelp in E flat."

I only desire here to put on record for historical reference the most respectable authorities of this day against Just Intonation, and to prove that the views I put forth have such opponents, and hence need to be told.

The interval 4 : 7 derived from the prime seventh has not been unnoticed, as a curiosity in acoustics; and it is occasionally referred to as the "Za" of Tartini. A living writer,* whose statements are entitled to the highest respect, and whose works contain most able arguments in favor of Just Intonation, says of the sounds produced from the prime seventh: "They may be called *anomalous.* They are wheels, but not wheels which will fit in with the previously constructed parts of the machine, and therefore they are left on one side."

The sound 4 : 7 has been known to be the seventh harmonic of the horn and æolian string, but has been called a "false" note, and has been rejected even by the advocates of just intonation, as opening the way for inextricable complication in theory and practice. It will from this, appear necessary to make the declaration which is the subject of this paper, and which is as follows:

The Prime Seventh belongs to the Musical System; its ratios are altogether appreciable by the common ear, and are in constant use in common music. It is this which constitutes, when added to the common chord, the *concord* (falsely called the *discord*) of *the Seventh,* and this element, combined with the other prime chords of Octave, Fifth, and Major-Third, makes the great variety of noble harmonies in which cultivated and uncultivated ears delight.

The prime seventh is necessary to complete the series of simple ratios, which extend as far as 10; and it was by noticing the blanks which its omission would leave that its necessity was discovered. The series is as follows:

1 : 2, 2 : 3, 3 : 4, 4 : 5, 5 : 6, [6 : 7, 7 : 8,] 8 : 9, 9 : 10

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* Gen. T. Perronet Thompson. Just Intonation. p. 72. 2d Edition. London, 1857. See also his "*Exercises, Political and Others.*" London. 1843. 6 vols. Both are in the Boston Athenæum.

or, if written as below, we shall have the natural series of harmonics, or what may be called the primary or

HARMONIC SCALE.

1 : 2 : 3 : 4 : 5 : 6 : 7 : 8 : 9 : 10.

As some reason should be assigned to the mathematician for not extending the series by the introduction of the *Prime Eleventh*, it will be found in the inability of the human ear to appreciate such complicated relations. The "*Chord of the Eleventh*" exists in nature, and I am able to tune it and to recognize its harmony in combinations specially made for the experiment; yet, so far as my examination of the works of the masters has extended, it has not been used by them in their written music, and perhaps never, unless possibly in the harmonics of a Paganini. I do not claim for it a place in our practical system of music, but leave it where all the former theorists have set the Prime Seventh. The Eleventh may hereafter be admitted, when the musical faculties of men have been sharpened by familiarity with the more simple concords *in their purity*, and when music is carried to a higher degree of refinement.

From the *Harmonic Scale* may be derived, by combination of its chords, an indefinite variety of other scales. The Octave is divided into eight intervals, which are convenient for melodic use, and the result is popularly called the Diatonic Scale, which, although generally taken as the basis, in explaining music, is not a primary, but a secondary Scale. The method of forming it, according to all former treatises, is, to take common chords (4 : 5 : 6) upon the tonic, dominant, and subdominant. Thus, the scale of C is tuned by taking a Fifth and Major-Third on C, on G, and on F, and bringing all the notes within the same octave.

But the introduction of the prime seventh allows of another division, in which only two fundamentals are employed; namely, the

Tonic (E) with its common chord, (C, E, G) and the Dominant (G), on which is taken the chord of seventh and ninth (G, B, D, F, A) in the ratios 4 : 5 : 6 : 7 : 9. To distinguish these scales, I have called the first the *Triple Diatonic*, and the last the *Double Diatonic*. Assuming the tonic or the key-note, as C, with 48 vibrations, the two scales will stand as follows: —

TRIPLE DIATONIC.

(With common Chord on C, on G, and on F.)

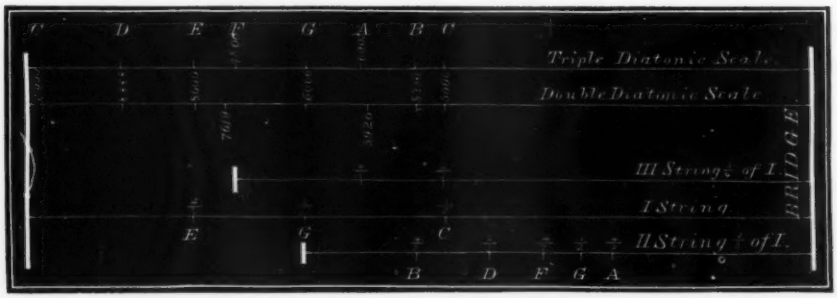
C	D	E	F	G	A	B	C
48	54	60	64	72	80	90	96
	8:9	9:10	15:16	8:9	9:10	8:9	15:16

DOUBLE DIATONIC.

(With common Chord on C: and Chord of 7 and 9 on G.)

C	D	E	(F)	G	(A)	B	C
48	54	60	63	72	81	90	96
	8:9	9:10	20:21	7:8	8:9	9:10	15:16

THE FOLLOWING DIAGRAM EXHIBITS THE MONOCHORD, WITH THE DERIVATION OF THE TWO DIATONIC SCALES, AND THE PROPORTIONAL LENGTHS OF THEIR STRINGS OR PIPES, THE PRIMARY HAVING 10000 EQUAL PARTS.



It thus appears that the fourth and sixth notes may be taken differently in intonation; and that this is done, can be easily observed by giving attention to singers. The Triple Diatonic has but three

different intervals; namely, 8:9*, 9:10 and 15:16. The Double Diatonic has, in addition, two others; namely, 20:21*, and 7:8; and in combinations its variety is greatly superior to the Triple Scale, whose chords and intervals are rather duplicates of one another.

And the remarkable fact is, that this Double Diatonic, which no theorist has defined, is more in practical use than the Triple, which stands in all the elementary books. A familiar example of the former is the "*O dolce Concento*" of MOZART, and the principal movement of the "*Dead March in Saul*" of HANDEL. The melody of the "*Hundredth Psalm*" is in the Triple Diatonic. The two scales often interchange, and an example of this is to be found near the close of "*O dolce Concento*," where for a single measure the dominant seventh and ninth yield, to admit the fourth and sixth of the Triple Scale.†

If it be feared that the distinctions which have been described as belonging to the scale will complicate it for those learning to sing or play, let it be added, that singers naturally observe them all, and need have no other instruction than to hear the sounds given by their teacher. What is here set down is of interest to him who wishes to know what is, and what ought to be, done. It may not be necessary for the singer to be even told the dimensions of any of his

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* In view of the numerous names required, and to supply those needed for these unnamed intervals, I have proposed (at least for mathematical and theoretical uses) names derived from the ratio. A fifth then will be "two-three," a major-tone "eight-nine," a diatonic semitone "fifteen-sixteen," the interval, (unnamed) between the third of the scale and the dominant seventh, is the "twenty-twenty-one." The next interval (dom. seventh to fifth) is the "seven-eight." This proposition as yet needs the approval of other theorists. The desideratum is accuracy and clearness.

† Not to disfigure these mathematical pages with musical types, I have chosen examples with which every one is familiar. Every composition will furnish others. If a choice is to be made in the examples of the profuse employment of the prime seventh, there may be taken any of the vocal scores of HAYDN, MOZART, or ROSSINI.

intervals; and it perhaps does no harm (except to the one who utters the falsehood) to say that all intervals are compounded of "semitones" or artificial twelfths of the octave.

It is true that when all the four primes have furnished their numerous chords and intervals, we shall have assembled a large number of notes, and it is not impossible that those unacquainted with music may fear that the number will be unmanageable, and prefer the compromises and limitations of temperament. As the experiment has been practically made, such persons may be assured that the musician can most easily produce his desired effects, when he has the full and abundant materials which the system of just intonation gives him.

The singers and players upon the free instruments, of their own accord, use the true intervals to the best of their ability; and in spite of the tempered instruments with which they are sometimes obliged to join. It is for men of science to indicate to the makers of imperfect instruments the way to perfect them; and to withhold their approval from players, who, from indolence or incapacity, only make a pretence of interpreting the music of the great masters. There are wanted no more apologies for, or speculations upon, the choice of temperaments; that subject has long ago been exhausted; and nothing more can be done than is now done with twelve tempered notes in the octave. When some economical astronomer shall propose to reduce the bulk and expense of the Nautical Almanac, by sacrificing that accuracy which gives it priceless value, the men at Greenwich will regard him as the scientific musician will, at a future day, look on those who would restrict him to the meagre and barbarous systems of temperaments of twelve notes.

NOTES AND QUERIES.

1. *Least Common Multiple.* On page 396, Vol. I., it is asked, whether the rule for finding the least common multiple, which says, divide by *any* number, will give the same result as the rule which says, divide by *any prime* number. My answer is, not always. If we divide the given numbers by *any prime* number which will divide two or more of them without a remainder, we shall, by following the rule, always get their *least* common multiple; if we divide by *any* number which will divide two or more of them, and follow the rule, we shall get a *common* multiple, but not always the *least.* The reason is plain; but may be most clearly seen by means of an example.

Find the least common multiple of 30, 63, 66, 300.

First Solution.

	30	63	66	300
6	2.3.5	$3^2.7$	2.3.11	$3.2^2.5^2$
5	5	$3^2.7$	11	2.5^2
	1	$3^2.7$	11	2.5

$6.5.3^2.7.11.2.5 = 207900$

Second Solution.

	30	63	66	300
2	2.3.5	$3^2.7$	2.3.11	$3.2^2.5^2$
3	3.5	$3^2.7$	3.11	$3.2.5^2$
5	5	3.7	11	2.5^2
	1	3.7	11	2.5

$2.3.5.3.7.11.2.5 = 69300$

The first solution gives a result three times too large, as it should. For when we divided by 6, the factor 3 common to all the numbers was taken out of only three of them. The least common multiple of several numbers is composed of all the factors not common to them, and of all the factors common to two or more of them, and of no others. Any common factor raised to the highest power to which it is found in either of the given numbers, must enter into the least common multiple, otherwise the supposed multiple would not be divisible by this number. On the other hand,

if any common factor, raised to a higher power than it is found in any of the given numbers, enters into the common multiple, then such multiple will not be the least, as is seen from the first solution; since 3 raised to the third power is found in the multiple 207900, which is one higher than it is found in either of the given numbers.—TEACHER.

2. On page 363, Vol. I., we find that 17 horses were devised to three sons, as follows: To the first, $\frac{1}{2}$; to the second, $\frac{1}{3}$, and to the third, $\frac{1}{9}$; and in the distribution by the Judge, the first got 9, the second 6, and the third 2. The question is, Was the distribution just?

Now it is evident that the testator intended to give the whole of the 17 horses to his three sons; and this he thought he had done; when, in fact, according to the terms of the will, he had only disposed of $\frac{17}{18}$ths of the horses, there being $\frac{1}{18}$ undisposed of. For $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$ only make $\frac{17}{18}$ths. But the rule is, to interpret wills so as to effectuate the *intention* of the testator, when that can be ascertained. In this case, it is plain he intended to give them *all* the horses, and named the proportions they should have. All that is necessary, then, to effectuate his intention, would be to say, As the whole portion of the horses actually devised to his three sons by the letter of the will is to the whole number of horses *intended* to be devised to them all, so is the particular proportion actually devised to any of the sons to the number or portion *intended* to be devised to that particular son.

Thus: $\frac{17}{18} : 17 :: \frac{1}{2} : 9 =$ first son's part.

$\frac{17}{18} : 17 :: \frac{1}{3} : 6 =$ second son's part.

$\frac{17}{18} : 17 :: \frac{1}{9} : 2 =$ third son's part.

So that the distribution is proved to be just, without resorting to the Judge's horse for aid in the interpretation of the will.

Or suppose a be the whole number of horses to be distributed. Then it will be as

$$\frac{17a}{18} : a :: \frac{a}{2} : \frac{9a}{17} = \text{the first son's part.}$$

$$\frac{17a}{18} : a :: \frac{a}{3} : \frac{6a}{17} = \text{the second son's part.}$$

$$\frac{17a}{18} : a :: \frac{a}{9} : \frac{2a}{17} = \text{the third son's part.}$$

So that a may be 17, or any number which is a multiple of 17. When that is the case, all the conditions ot the question can be complied with, so far as the distribution is concerned, by multiplying the number of horses by 9, and dividing by 17, for the first son's part; by multiplying by 6, and dividing by 17, for the second son's part; and multiplying by 2, and dividing by 17, for the third son's part. — Saxe Gotha Laws, Attorney and Counsellor at Law, Dover, Delaware.

3. *Note on Maxima and Minima.* The following deduction of the rule for determining maxima and minima values of a function may aid the student of the Calculus to a better understanding of the common symbolic demonstration.

A function has a maximum or minimum value when its rate of change, or the first derivative, vanishes, and in vanishing, changes its sign.

Since a positive value in the derivative indicates an increase of the function for an increase of the variable, and a negative value indicates a decrease of the function for an increase of the variable; then a change in the derivative from positive to negative values indicates a change in the function from increase to decrease, and hence the value of the function is a maximum.

On the other hand, a change in the derivative from negative to positive values indicates a change in the function from decrease to increase, and the function has a minimum value.

There are four ways in which a continuous real quantity may vanish: 1st, by decreasing from positive to negative values; 2dly, by increasing from negative to positive values; 3dly, by having zero as a minimum value; and 4thly, by having zero as a maximum value. These four states of zero may be expressed by the symbols $+0-$, $-0+$, $+0+$, and $-0-$; in which the signs indicate the states of the quantity before and after vanishing.

If a derivative has, for any value of the variable, the value $+0+$, or $-0-$, then the function increases or decreases both before and after it has ceased to increase or decrease; that is, it simply suspends for a moment without reversing the change, and its value is therefore neither a maximum nor a minimum.

If a series of successive derivatives all vanish for a given value of the variable; and if the following one which does not vanish, have a positive value, then the last of the vanishing derivatives is zero by increase, and is therefore of the form $-0+$. The last but one is thence a minimum zero, or $+0+$; the last but two is $-0+$, the same as the last of the series. The next lower is, therefore, $+0+$, and so on.

On the other hand, if the value of the derivative which does not vanish be negative, then the last of the series will be $+0-$, the last but one $-0-$, the last but two $+0-$, and so on.

If we denote by D^n the derivative following the vanishing series, and by D^{n-1}, D^{n-2}, &c., the derivatives of the vanishing series, these propositions may be exhibited as follows:

If	$D^n = +a$	or $-a$
then	$D^{n-1} = -0+$	or $+0-$
	$D^{n-2} = +0+$	or $-0-$
	. . . . .	
	$D^{n-2i} = +0+$	or $-0-$
and	$D^{n-2i-1} = -0+$	or $+0-$

Hence if D^{n-2i} be the first derivative of the function, n must be $2i + 1$, an odd number, and the value of the function is neither a maximum nor a minimum, since it does not reverse its rate of change. But if D^{n-2i-1} be the first derivative n must be $2i+2$, an even number, and the value of the function is a minimum when $D^n = +a$, and a maximum when $D^n = -a$. Hence the rule.

If the derivative, the first that does not vanish for a given value of the variable, be positive, and of an even order, the function is at a minimum value; if this derivative, be negative and even, the function is at a maximum value; but if this derivative be of an odd order, then the value of the function is neither a maximum nor a minimum. W.

ON THE INDETERMINATE ANALYSIS.

By Rev. A. D. Wheeler, Brunswick, Maine.

DEFINITIONS.

1. That branch of Algebra which treats of the solution of indeterminate or unlimited problems is denominated The Indeterminate Analysis.

2. An Indeterminate Equation is one that contains two or more unknown quantities.

3. The Diophantine Analysis is that in which the indeterminate quantities are required to be of the second, or of some higher degree.

4. The Factors of a number are any numbers which, multiplied together, will produce it; or in other words, they are any of its exact divisors.

5. Prime Numbers are those which have no factor, excepting unity and the numbers themselves.

6. The Prime Factors of a number are its least divisors.

7. Numbers are said to be prime to each other, when they have no common divisor.

8. All numbers that are not prime are termed composite numbers.

9. An Integer is a number composed of entire units.

AXIOMS.

1. If an integer be added to an integer, the sum will be an integer.

2. If an integer be subtracted from an integer, the remainder will be an integer.

3. If an integer be multiplied by an integer, the product will be an integer.

4. If an integer be divided by any of its factors, the quotient will be an integer.

PROPOSITIONS.

I. If a number will divide the sum of two or more numbers, and all the parts excepting one, then will it also divide the remaining part.

DEMONSTRATION. Let $S = A + B$. Then if d be an exact divisor of S and A, it will also be an exact divisor of B. For if we supposed d to be contained n times in S, and n' times in A, then will it obviously be contained $n - n'$ times in B; and this must be an integer according to Axiom 2.

In the same way may the truth of the proposition be shown, whatever be the signs, or the number of parts.

II. If a number will divide all the parts of any number ex cepting one, and will not divide that one, then it will not divide their sum.

Demonstration. If it would divide their sum, then, according to what has just been proved, it would likewise divide all the parts; which is contrary to the supposition.

III. If P be a prime number and will divide the product, $A.B$, then will it divide one of the factors A, or B.

Demonstration. Since P is a factor of the product, $A.B$, it must be found among the factors of either A or B, and consequently it must divide one or the other.

IV. If P will divide neither A nor B, it will not divide their product.

Demonstration. If it would divide their product, then, according to the last proposition, it must divide A or B; which is contrary to the supposition.

V. If a and b are prime to each other, and if, in the expression $\frac{by}{a}$, we assign to y every possible integral value less than a, and positive, the expression can never become an integer, and in no two cases shall we have the same remainder.

Demonstration. (1) As a is prime to b, and greater than y, it will divide neither factor, and consequently will not divide their product, Prop. IV.

(2). Suppose that any two values of y, as y' and y'', will give the same remainder r, then $\frac{by'-r}{a}$, and $\frac{by''-r}{a}$, will both be integers. Hence, also, their difference $\frac{by'-r}{a} - \frac{by''-r}{a} = \frac{b(y'-y'')}{a}$, (Axiom 2) is an integer. But as $y' - y''$ is less than y', which is, by hypothesis, less than a, it is plain that a will divide neither factor, and therefore it is impossible that $\frac{b(y'-y'')}{a}$ should be an integer. Consequently no two remainders can have the same value.

Cor. 1. Since the remainders are all different, and the num-

ber of them is equal to $a-1$, they must evidently comprehend the entire series of natural numbers between the limits 0, and a. Let any one of these remainders be r; then $\frac{by-r}{a}$ will be an integer; or let it be $a-r$; then $\frac{by-(a-r)}{a} = \frac{by-a+r}{a} = -1 + \frac{by+r}{a}$ is an integer. Therefore, $\frac{by+r}{a}$ is an integer. Prop. I.

Hence the expression $\frac{by \pm r}{a}$ is always possible in positive whole numbers.

Cor. 2. Let c represent any constant quantity. Then will the expression $\frac{c \pm by}{a}$, under like circumstances, give different remainders. For whatever remainder results from the division of c by a, this also will be constant; and therefore whether we add to it or subtract from it, the unequal remainder, resulting from the division of by by a, the sums and the differences will likewise be unequal.

Hence, also, the expression $\frac{c \pm by}{a}$ is always possible in integers, either positive or negative, according to the relations between c and by.

REMARKS.

1. If, in the last proposition, we give to y a value equal to a, or any multiple of a; the expression becomes $\frac{bma}{a} = bm$, leaving no remainder.

2. If we make $y = ma+1, ma+2$, &c., we shall have $\frac{bma+b}{a}$, $\frac{bma+2b}{a}$, &c., equal to $\frac{by}{a}$; from which we shall obviously obtain the same remainders as when we give to y the successive values, 1.2.3. &c.

Whence it appears, that after y becomes a, or any multiple of a, the remainders will begin to return in the same order.

3. If a and b are not prime to each other, let m' be a common factor. Then we have $\frac{by}{a} = \frac{m'b'y}{m'a'} = \frac{m'}{m'} \times \frac{b'y}{a'}$, and the periods of the remainders will begin to return when $y = a'$, or ma', and will all be increased by the same multiplier m.

4. If in the expression $\frac{c \pm by}{a}$ we suppose a to be a composite number, as ma', the expression becomes $\frac{c \pm by}{ma'}$. Now when b is a multiple of either m or a', and y becomes a multiple of the other, the period of remainders, upon performing the division, will begin to return after a' terms, when b is a multiple of m, and after m terms, when it is a multiple of a'.

First, suppose that b is a multiple of m; then we have $\frac{c \pm by}{a} = \frac{c \pm mb'm'a'}{ma'} = \frac{c}{ma'} \pm b'm'$, or $\frac{c}{a} \pm b'm'$; from which will result the original remainder of c divided by a.

Again, suppose that b is a multiple of a'; then we have in the same way, $\frac{c \pm by}{a} = \frac{c \pm a'b'my'}{ma'} = \frac{c}{ma'} \pm b'y' =$ or $\frac{c}{a} \pm b'y'$, which gives the same remainder as before.

All the preceding propositions may be readily illustrated by the substitution of numbers.

[To be continued.]

ON THE COMPRESSIBILITY OF LIQUIDS.

By Pliny Earle Chase, Philadelphia, Pa.

Let d = density, or specific gravity of a liquid (the specific gravity of air being 1).

If air is a perfectly elastic fluid, it would require, by the law of Mariotte, a pressure of d atmospheres to reduce it to the density d. By the same law, $\frac{1}{d+1}$ = additional compressibility of air at

density d, by 1 additional atmosphere; $d\left(\frac{1}{d}-\frac{1}{d+a}\right)=\frac{a}{d+a}=$ additional compressibility of air at density d, by a additional atmospheres.

It seems reasonable to suppose that the ratio of compressibility between the liquid and air of the same *density* may be some function of $d\times$ the compressibility of air under the same *pressure*. This function may perhaps be represented by the ratio of homologous representatives of equal weights of air and liquid under the ordinary atmospheric pressure; for example, by the ratio of the diameter of a sphere of air or liquid at density d, to the diameter of the same sphere at density 1, which is the ratio of $\frac{1}{d^{\frac{1}{3}}}$. If such is the case, $\frac{1}{d^{\frac{1}{3}}(d+1)}\times\frac{1}{2}$*$=$ compressibility of liquid by 1 additional atmosphere.

The assumption of the specific gravity of air as the unit of the ratio, is undoubtedly, to some extent, arbitrary. But as soon as the pressure of the air is removed, in any degree, from the surface of a liquid, a portion of the liquid assumes the form of vapor, and it would be impossible to approximate so nearly to the average density of the mixed liquid and vapor, at any other density of the air, as we can at the one we have assumed. It is true that even at the ordinary pressure of the atmosphere, liquids are constantly evaporating; but as that pressure is appointed by the Creator, as the one of natural equilibrium, it seems reasonable to presume that we shall not materially err, in taking it as the starting point of our hypothesis. To test that hypothesis, let us examine

Water, which has	$d=812.5$	Mercury, which has	$d=11023.7$
Alcohol, " "	$d=642.7$	Sulphuric Ether, " "	$d=581.7$

* $\frac{1}{2}=$ compressibility of air under pressure of a atmospheres, by the pressure of a additional atmospheres.

According to the above formula, the compressibility by a single additional atmosphere, would be, for

Water, $\frac{1}{\sqrt[3]{812.5 \times 813.5}} \times \frac{1}{2} = .00006586.$

Alcohol, $\frac{1}{\sqrt[3]{642.7 \times 643.7}} \times \frac{1}{2} = .00009001.$

Mercury, $\frac{1}{\sqrt[3]{11023.7 \times 11024.7}} \times \frac{1}{2} = .00000203.$

Sulph. Ether, $\frac{1}{\sqrt[3]{581.7 \times 582.7}} \times \frac{1}{2} = .00010279.$

In the following Table, the theoretical are compared with different experimental results.

Compressibility in Millionths, by 1 Atmosphere.	THEORY.	COLLADON and STURM.	DANIELL.	OERSTED.	CANTON.	PERKINS.
Of Water,	65.86	49 to 51	46.65	46	40 to 46	48 to 100
" Alcohol,	90.01	93 to 96	21.65	20	66	
" Mercury,	2.03	5	2.65	1	3	
" Sulph. Ether,	102.79	133 to 150	61.65	60		

The experimental results are undoubtedly all subject to correction for friction, imperfections of apparatus, and difficulty of minute observation; and the theoretical results should likewise be modified by considerations of temperature, vapor, imperfect elasticity of the air, and perhaps other unknown influences. Our data are so few and imperfect, that they cannot be regarded as deciding the truth or falsity of the theory. They may, however, be sufficient to call attention to an almost untrodden field of investigation, awakening an interest which will be followed by more numerous and accurate experiments, and lead to the modification of the formula here suggested, in the discovery of a new one, which will more accurately represent the law of fluid compressibility. It is already known that the law of MARIOTTE is only approximately true. When the compressibility of various different gases and vapors has been determined with the greatest possible

precision, we may, perhaps, theoretically determine, with equal precision, the compressibility of the condensed vapors in the liquid form.

The first experiments which proved with any tolerable degree of accuracy the compressibility of water were those of PERKINS. The results of his early trials gave a degree of compressibility more than twice as great as his final estimate; but the ratios of compression between the successive experiments correspond very nearly, in many instances, with the theoretical ratios. I can hardly believe that this correspondence is entirely accidental.

In the following table, the first column gives the number of atmospheres (a) to which a column of water, 190 inches in length, was subjected; second column, the compressibility, in decimals of an inch, according to PERKINS's table, with the ratios between the results of the successive experiments; third column, the theoretical compressibility in decimals of an inch, with the successive ratios. The numbers in the third column are obtained by the formula $190 \times \frac{a}{d^{\frac{1}{4}}(d+a)} \times \frac{1}{2}$.*

PRESSURE.	Compressibility according to PERKINS.	Theoretical Compressibility.	PRESSURE.	Compressibility according to PERKINS.	Theoretical Compressibility.
1 atmosph.	.019	.0125	150 atm's.	1.914	1.5866
	9.94	9.90		1.27	1.27
10 "	.189	.1238	200 "	2.440	2.011
	1.97	1.97		1.37	1.37
20 "	.372	.2438	300 "	3.339	2.7454
	1.47	1.48		1.26	1.22
30 "	.547	.3625	400 "	4.193	3.3587
	1.26	1.32		1.21	1.16
40 "	.691	.4777	500 "	5.087	3.8784
	1.17	1.24		1.16	1.11
50 "	.812	.5902	600 "	5.907	4.3247
	1.18	1.18		1.14	1.09
60 "	.956	.7001	700 "	6.715	4.7117
	1.10	1.15		1.10	1.07
70 "	1.056	.8075	800 "	7.402	5.051
	1.12	1.13		1.11	1.06
80 "	1.187	.9126	900 "	8.243	5.3505
	1.09	1.11		1.09	1.05
90 "	1.288	1.0152	1000 "	9.002	5.617
	1.10	1.10		1.76	1.29
100 "	1.422	1.1157	2000 "	15.833	7.2394
	1.35	1.42			

* 190 = length of experimental column; a = number of atmospheres; $d = 812.5$.

THE ELEMENTS OF QUATERNIONS.

By W. P. G. Bartlett, Cambridge, Mass.

The following brief essay is intended to present, in a direct method, the fundamental principles and notation of the *Quaternion Analysis;* so as to enable one who has mastered it to proceed at once, without difficulty, to the geometrical applications and further developments given in the VIIth of Hamilton's "*Lectures on Quaternions*" or elsewhere. With a few trifling exceptions, Hamilton's notation is strictly retained. Nothing will be given, at present, upon the *differentials of quaternions*, which are introduced in Lecture VII., § xcvii., of Hamilton.

The author intends hereafter to take up, as a continuation of these papers, some special subject, such as Spherical Trigonometry, or Analytic Geometry, and discuss it in the language of Quaternions.

I. — Lines.

1. *Equal* lines are such as have the same length and the same direction in space.

2. One line is the *negative* of another when it has the same length, but the opposite direction.

3. The *sum* of two lines is the diagonal of a parallelogram, of which these lines are two adjacent sides; that diagonal being taken, whose direction lies between the directions of the two given lines. Hence the sum of two or more lines is the same, in whatever order they may be placed or added together; and when any set of lines in space form a closed circuit, their sum is equal to zero, and therefore each line is the negative of the sum of all the others.

4. In operations on lines in space, it is convenient to substitute for some of the given lines other lines, equal, and therefore parallel, to the given ones, but passing through a point in space com-

mon to themselves and the other given lines; so that all the lines may be coöriginal. In what follows this is supposed to be the case.

II. — Quaternions.

5. The abstract operation of so changing the length and direction of one line as to make it coincide in length and direction with another line, is considered as the *quotient* of the second line *divided* by the first, and is called a *quaternion*. The plane in which, by § 4, both these lines are situated, is called the *plane* of the quaternion. Conversely the *product* of *multiplying* the first line by the quaternion is the second line. In writing a product the *multiplier* always precedes the *multiplicand*.

6. Lines are generally denoted by the letters α, β, γ, &c.; quaternions by p, q, r, &c. The angular distance in the plane of α and β from the positive direction of α to that of β is denoted by $\frac{\beta}{\alpha}$.

7. The quotients $\beta \div \alpha$, and $\delta \div \gamma$, are *equal*, when α, β, γ, and δ being coplanar, $\frac{\beta}{\alpha} = \frac{\delta}{\gamma}$, and the ratio of the length of β to that of α is equal to ratio of the length of δ to that of γ. Hence if there are given any two quaternions, $q = \beta' \div \alpha'$, and $p = \gamma' \div \delta'$, there can always be found three other lines, α, β, and γ, such that $\beta \div \alpha = \beta' \div \alpha' = q$, or $\beta = q\,\alpha$; and $\gamma \div \alpha = \gamma' \div \delta' = p$, or $\gamma = p\,\alpha$; that is, some line, α, can always be found such that it may be multiplied by both the given quaternions. The three lines α, β, and γ might have been so determined that $\beta \div \alpha = \beta' \div \alpha' = q$, and $\gamma \div \beta = \gamma' \div \delta' = p$.

8. The *sum* of two quaternions is defined by the equations

$$p = \beta \div \alpha,\ q = \gamma \div \alpha,\ p + q = \beta \div \alpha + \gamma \div \alpha = (\beta + \gamma) \div \alpha;$$

or,
$$p\,\alpha = \beta,\ q\,\alpha = \gamma,\ (p + q)\,\alpha = \beta + \gamma = p\,\alpha + q\,\alpha.$$

By §§ 3 and 5, $p + q = (\beta + \gamma) \div \alpha$ must be a quaternion. Hence

$\Sigma q = q'$; that is, the sum of two or more quaternions is a quaternion; and, by § 3, it is the same in whatever order its component quaternions may be taken.

9. The *product* of multiplying q by p is defined by the equations

$$q = \beta \div \alpha,\ p = \gamma \div \beta,\ pq = \gamma \div \alpha\ ;$$

or,
$$q\alpha = \beta,\ p\beta = \gamma,\ pq\,.\,\alpha = p\,.\,q\alpha^* = p\beta = \gamma.$$

From its form it follows, as in § 8, that $\Pi q = q'$; that is, the product of two or more quaternions is a quaternion. If $pq = r$, we may write $p = r \div q$, which defines the *quotient* of two quaternions. But we cannot write $q = r \div p$, unless $qp = r$; and it must be observed that a quaternion is not, in general, *commutative;* that is, independent of the order of its factors; for, let the additional lines δ and ε be so determined that $\delta \div \beta = \beta \div \alpha = q$, and $\beta \div \varepsilon = \gamma \div \beta = p$; whence $qp = \delta \div \varepsilon$; but by § 7, δ must be in the plane of α and β, and ε in that of β and γ; therefore δ and ε cannot be in the plane of α and γ, unless β is also in this plane; so that qp is different from pq, unless p and q are *coplanar.*

(To be continued.)

REPORT OF THE JUDGES ON PRIZE ESSAYS.

Only three essays were presented for our consideration; one on Central Forces; one on Spherical Conics; and one on Projectiles. We are of opinion that the first alone has sufficient merit to entitle it to a prize; and to this, as the "best essay," we award the prize of Fifty Dollars. Prof. Perkins was unable to concur in this report.

William Ferrel.

September 10th, 1859. John B. Henck.

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* Points are used, in this case, to separate a product into its proper factors; thus, this equation may be read pq times $\alpha = p$ times $q\alpha$.

Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the July number of the MONTHLY: — GUSTAVUS FRANKENSTEIN, Springfield, Ohio; F. T. HAMPTON, Columbian College, Washington, D. C.; ROLAND THOMPSON, Junior Class, Jefferson College, Canonsburg, Penn.; OTHO E. MICHAELIS, Freshman Class, Free Academy, N. Y.; DAVID TROWBRIDGE, Perry City, Schuyler Co., N. Y.; J. T. LOVEGROVE, Baltimore, Md.

It gives us pleasure to announce that the well-known and enterprising firm of IVISON & PHINNEY have become the publishers of the MATHEMATICAL MONTHLY. We are satisfied that their energy and large business facilities not only guarantees the permanency of the MONTHLY, but a wider usefulness and a larger measure of success. In advance of a fuller announcement, Messrs. IVISON & PHINNEY authorize us to say, that the continued issue of the MATHEMATICAL MONTHLY is now placed beyond any contingency; that it will continue to be manufactured in Cambridge, under the supervision of the Editor; that its present character of being among the finest specimens of mathematical printing ever executed, will be fully maintained; that the size of the second and succeeding volumes will not fall below Vol. I.; that each volume will contain at least two steel plates, not inferior in artistic excellence to those illustrating DONATI'S Comet, in Vol. I.; and lastly, that the MONTHLY will be enlarged and improved just as fast as its circulation will warrant. They propose to issue a largely increased edition, and bring it to the attention of all the prominent professors, teachers, engineers, &c., in the country at all likely to be interested in such a publication. The Editor would improve the opportunity to say, for the benefit of those who may never have seen the first number of Vol. I., in which he has developed pretty fully the plan and aims of the MONTHLY, that its primary aim is educational, and it is designed to meet the wants of teachers and students generally. At the same time it will embrace in its pages investigations in the higher mathematics, and thus secure the coöperation of those who are enlarging the boundaries of this science. We earnestly invite the attention of all to the claims of the MONTHLY, and hope that, during the coming year, the number of those who are now coöperating with us to secure the ends of its establishment, may be largely increased. All contributions should still be addressed to the Editor, whose editorial relations to the MONTHLY are entirely unchanged.

OUR PRIZES. The success of these prizes during the past year, fully warrants their continuance, with the following modifications: — In each number Five Prize Problems will, as heretofore, be published, of which the first and second will be quite elementary; and for the best solution of which a prize of a copy of the MONTHLY is offered, by Messrs. IVISON & PHINNEY. This prize is open to all students in Academies, High Schools, and all institutions, whether public or private, not conferring degrees. The successful competitor will receive the volume free. The remaining three of the five problems will be open to the same competition as heretofore, with the following prizes: For the best solutions we offer a first prize of six dollars, and a second prize of four dollars to those of the second order of merit. The prize solutions will continue to be published. Judges: SIMON NEWCOMB, Esq., W. P. G. BARTLETT, Esq., and TRUMAN HENRY SAFFORD, Esq. . . . . The prizes for essays remain the same, and are offered under the same conditions, except that the first and second are open to all competitors; and it shall be discretionary with the judges to decide to which of the prizes the best essay, the second in order of merit, &c., are entitled. Judges: WILLIAM FERREL, Esq., J. B. HENCK, Esq., and CHAUNCEY WRIGHT, Esq. Mr. WRIGHT takes the place of Prof. PERKINS, who declines to serve another year. See pp. xi. and xii., No. for October, Vol. I., for conditions.

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Among the Testimonials recently received, are strong commendations from Prof. HAYES, State Essayer of Massachusetts; Prof. EVANS, Marietta College; Prof. HILLMAN Dickinson College; Dr. WEBSTER, Principal of the New York Free Academy; the Rev. D. C. VAN NORMAN, New York; Prof. BARNES, United States Naval Academy, Annapolis, Md.; President COLLINS, Dickinson College; Prof. CLEVELAND, Philadelphia; Prof. HOPKINS, Genesee College; THOMAS SHERWIN, A. M., Principal of English High School, Boston, and many others; as also cordial commendations from the Tribune, New York Observer, Boston Journal, Boston Recorder, Evangelist, Churchman, New York Teacher, Educational Herald, School and Home Journal, Scientific American, The Independent, Life Illustrated, Christian Advocate and Journal, Examiner, Christian Inquirer, Democratic Review, Publishers' Circular, Boston Traveller, Post, and Journal, &c., &c.

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[SEE ALSO LIST ON NEXT PAGE.]

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